Khovanov homology via immersed curves

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Khovanov homology

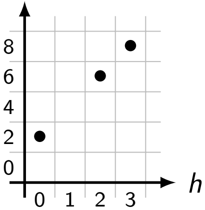
$$\text{Knot } K \xmapsto{\text{[Khovanov'00]}} \boxed{\mathrm{Kh}(K)} \ \underset{h}{\mathbb{Z}} \oplus \underset{q}{\mathbb{Z}}\text{-graded } \mathbf{k}\text{-vector space}$$

- Combinatorial construction
- $\mathrm{Kh}(K)$ categorifies Jones polynomial
- $\widetilde{\mathrm{Kh}}(K)$ is the reduced version
- Examples: $\widetilde{\mathrm{Kh}}($$) = \mathbf{k}$, $\widetilde{\mathrm{Kh}}($$) = \mathbf{k}^3$, $\widetilde{\mathrm{Kh}}($$) = \mathbf{k}^5$
- Major application: integer-valued concordance invariant $s(K)$, used to combinatorially prove $u(T(p,q)) = \frac{1}{2}(p-1)(q-1)$ [Rasmussen'04]

Bar-Natan homology (aka S^1-equivariant Khovanov homology)

Khovanov homology

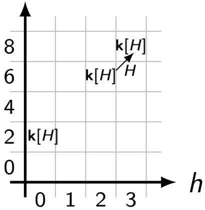
$\widetilde{\mathrm{Kh}}($$) = \mathbf{k}^3$

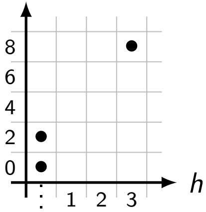


Bar-Natan homology [Bar-Natan'05]

- working over $\mathbf{k}[H]$
- differential is deformed

$\widetilde{\mathrm{BN}}($$) = \mathbf{k}[H] \oplus \mathbf{k}[H]/(H)$





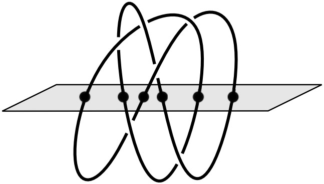
Remark: The two are connected via mapping cone:

$$\widetilde{\mathrm{Kh}}(K) = \mathrm{H}_*[\widetilde{\mathrm{BN}}(K) \xrightarrow{H} \widetilde{\mathrm{BN}}(K)]$$

Symplectic Khovanov homology

- Kh(K) defined combinatorially $\implies$ Q. Are there geometric / topological viewpoints on Khovanov homology?
- Algebro-geometric interpretation [Cautis-Kamnitzer'08]
- Gauge-theoretic proposal [Witten'16]
- Floer-theoretic interpretation of Kh($K;\mathbb{Q}$) [Seidel-Smith'06, Manolescu'06, Abouzaid-Smith'16'19]

  Input: n-bridge decomposition of a knot



$$T_1 \longmapsto L(T_1) = (S^2)^n$$
$$\downarrow$$
$$\mathcal{S}_{2n} \longmapsto \mathcal{M}(\mathcal{S}_{2n})^{4n}$$
$$\uparrow$$
$$T_2 \longmapsto L(T_2) = (S^2)^n$$

Output: $\mathrm{Kh}(K) \cong \mathrm{HF}(L(T_1), L(T_2))$

Shifting gears

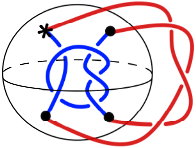
Natural question: what are the implications of symplectic Khovanov homology?

One answer: geometric approach towards equivariant Khovanov homology [Seidel-Smith'10, Hendricks-Lipshitz-Sarkar'15].

Difficulty: the moduli space $\mathcal{M}(\mathcal{S}_{2n})$ is complicated.

So let's change the perspective:

- Assume that $n = 2$ in $K = T_1 \cup_{\mathcal{S}_{2n}} T_2$
- Do <u>not</u> require T_1, T_2 to be trivial



Goal: interpret $\widetilde{\mathrm{BN}}(K)$ and $\widetilde{\mathrm{Kh}}(K)$ as wrapped Floer homology of immersed curves in $\mathcal{S}_4$.

Remark: $\dim(\mathcal{S}_4) = 2$ is lower than 8 or 4.
Reason: we work on the reduced S^1-equivariant level.

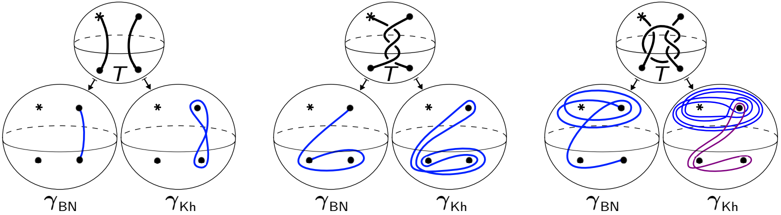
Curve invariants

Input: pointed 4-ended tangle $T\colon I^* \sqcup I \hookrightarrow D^3$. E.g. 

Output: two tangle invariants $\gamma_{\text{BN}}(T)$, $\gamma_{\text{Kh}}(T)$, each a collection of bigraded oriented immersed curves with local systems on $\mathcal{S}_4^* = \partial(D^3, T)$.

Construction: later.

Examples:

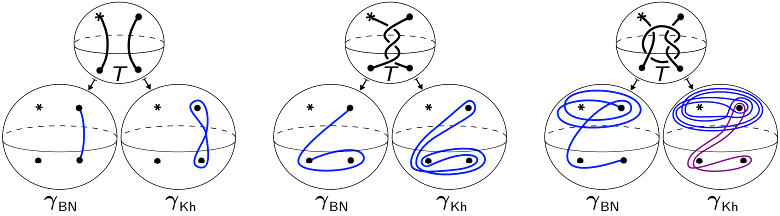


(a) Trivial tangle. (b) Three-twist tangle. (c) (2,-3)-pretzel tangle.

Properties of curve invariants

1. $\gamma_{\mathrm{BN}}(T)$ is of the form $\{$one arc$\} \cup \{$compact curves$\}$
2. $\gamma_{\mathrm{Kh}}(T)$ consists of only $\{$compact curves$\}$
3. MCG naturality in char. two: $\sigma(\gamma_{\mathrm{BN}}(T;\mathbb{F}_2)) = \gamma_{\mathrm{BN}}(\sigma(T);\mathbb{F}_2)$

   $\implies$ for rational tangles:
   - $\gamma_{\mathrm{BN}}(R;\mathbb{F}_2)$ is obtained from the tangle $R\colon\; I^* \sqcup I \hookrightarrow D^3$ by pushing the unmarked component I into the boundary sphere
   - $\gamma_{\mathrm{Kh}}(R;\mathbb{F}_2)$ is obtained by substituting $\gamma_{\mathrm{BN}}(R;\mathbb{F}_2)$ by a figure eight
4. Curve invariants do depend on the field **k**



(a) Trivial tangle. (b) Three-twist tangle. (c) (2,-3)-pretzel tangle.

Main theorem

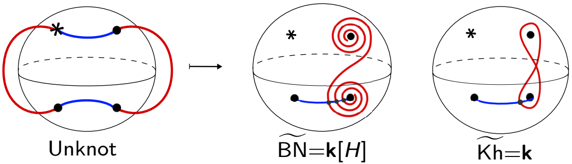
Theorem (K-Watson-Zibrowius'19)

Suppose a knot K is decomposed along a pointed 4-punctured sphere, that is $K = T_1 \cup_{\mathcal{S}_4^*} T_2$. Then its reduced Bar-Natan homology and reduced Khovanov homology (as bigraded **k***-vector spaces) are isomorphic to wrapped Floer homology of the curves associated to the two tangles:*

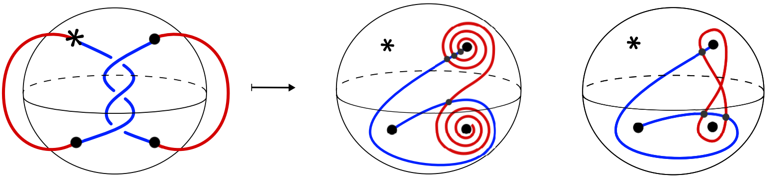
$$\widetilde{\mathrm{BN}}(K) \cong \mathrm{HF}\left(\gamma_{\mathrm{BN}}(T_1), \gamma_{\mathrm{BN}}(T_2)\right)$$

$$\widetilde{\mathrm{Kh}}(K) \cong \mathrm{HF}\left(\gamma_{\mathrm{Kh}}(T_1), \gamma_{\mathrm{BN}}(T_2)\right)$$

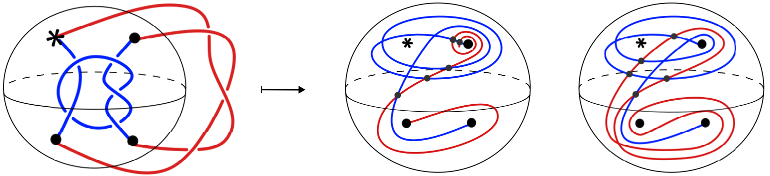
Example:



More examples



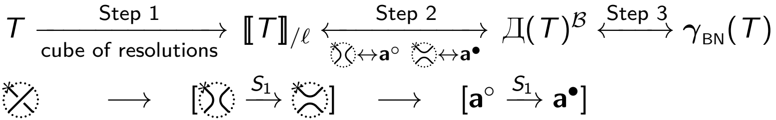
Trefoil | $\widetilde{\mathrm{BN}}=\mathbf{k}[H]\oplus\mathbf{k}[H]/(H)$ | $\widetilde{\mathrm{Kh}}=\mathbf{k}^3$



(4,-3)-torus knot | $\widetilde{\mathrm{BN}}=\mathbf{k}[H]\oplus\mathbf{k}[H]/(H^2)\oplus k[H]/(H)$ | $\widetilde{\mathrm{Kh}}=\mathbf{k}^5$

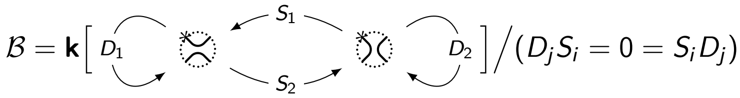
Remark: third type of immersed curve $\mapsto$ unreduced $\mathrm{Kh}(K)$ is obtained via Floer homology as well

Construction of $\gamma_{BN}(T)$



Step 1. 4-ended tangle $T \xmapsto{\left[\substack{\text{Bar-Natan'05,}\\ \text{Khovanov'02, Manion'17}}\right]} [\![T]\!]_{/\ell}$ $\left(\substack{\text{considered}\\ \text{up to homotopy}}\right)$

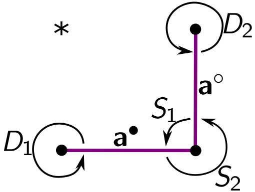
<u>Proposition.</u> $[\![T]\!]_{/\ell}$ is a twisted complex over *deformed reduced* arc algebra, described by the quiver below:



Step 2. <u>Key observation:</u> $\boxed{\mathcal{B} \hookrightarrow \mathcal{W}(\mathcal{S}_4^*)}$ via

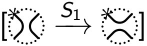
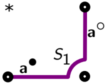
$\mathcal{B} \cong \mathrm{End}_{\mathcal{W}(\mathcal{S}_4^*)}(\mathbf{a}^\circ \oplus \mathbf{a}^\bullet) = \bigoplus_{i,j \in \{\circ,\bullet\}} \mathrm{CF}(\mathbf{a}^i, \mathbf{a}^j)$

Note: here we use the choice of $*$.



Construction of $\gamma_{\text{BN}}(T)$ + pairing theorem

$$T \xrightarrow[\text{cube of resolutions}]{\text{Step 1}} [\![T]\!]_{/\ell} \xleftrightarrow[\text{(resolution)}\leftrightarrow \mathbf{a}^{\circ}\ \ \text{(resolution)}\leftrightarrow \mathbf{a}^{\bullet}]{\text{Step 2}} Д(T)^{\mathcal{B}} \xleftrightarrow[\text{[HKK]}]{\text{Step 3}} \gamma_{\text{BN}}(T)$$

 $\longrightarrow$  $\longrightarrow$ $[\mathbf{a}^{\circ} \xrightarrow{S_1} \mathbf{a}^{\bullet}]$ $\longrightarrow$ 

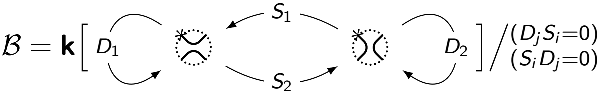
Step 3. $Д(T)^{\mathcal{B}} \longleftrightarrow \gamma[Д(T)] = \gamma_{\text{BN}}(T)$

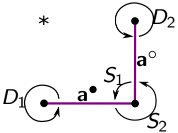
<u>Theorem.</u> (Haiden-Katzarkov-Kontsevich'14) Homotopy classes of twisted complexes over the wrapped Fukaya category of a surface are in 1-1 correspondence with immersed curves equipped with local systems.

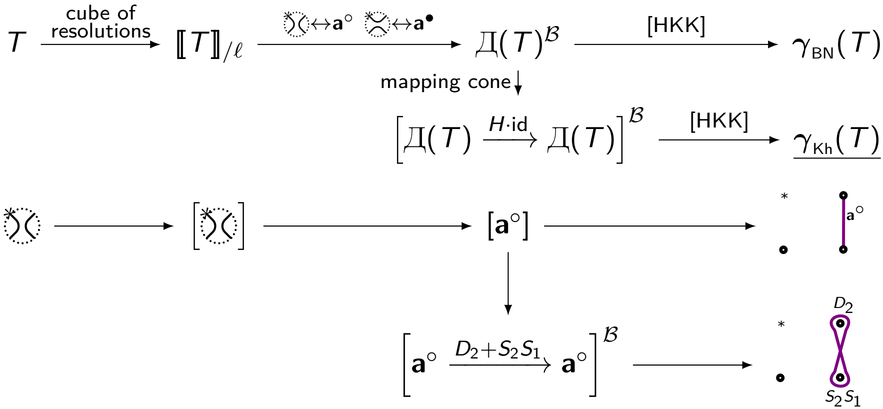
- We reproved it geometrically, using the train-track approach from [Hanselman-Rasmussen-Watson'17]
  $\Longrightarrow$ curve invariants are computable [Zibrowius-Chhina'19]
- Further classified morphism spaces:
  $\text{Mor}(N_1^{\mathcal{B}}, N_2^{\mathcal{B}}) \simeq \text{CF}(\gamma[N_1], \gamma[N_2]) \Longrightarrow$ pairing theorem

Construction of γ_{Kh}

Define a central element $\boxed{H := D_1 + D_2 + S_1S_2 + S_2S_1}$ in



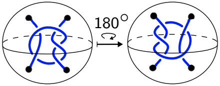




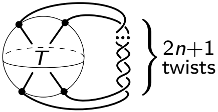
Tangle replacement questions

Goal: apply immersed curves to tangle replacement questions.

Conjecture 1.
Kh$(K;\mathbb{Q})$ is preserved by mutation.



Conjecture 2. (Generalized Cosmetic Crossing Conjecture)
Assuming the equator on the right is not compressible, all the knots in the family $K_n = T(2n+1)$ are different.

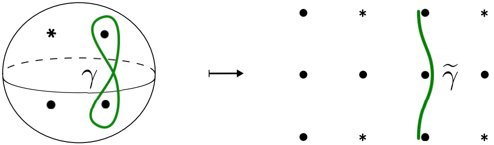


The key is to study **the geography question**:

Which immersed curves can be components of $\gamma_{\mathrm{Kh}}(T)$*?*

Geography of curve invariants

- Consider the cover $(\mathbb{R}^2 \smallsetminus \mathbb{Z}^2) \to (T^2 \smallsetminus 4\text{pt}) \xrightarrow{\text{quotient by the elliptic involution}} \mathcal{S}_4^*$
- Will study immersed curves $\gamma \looparrowright \mathcal{S}_4^*$ via their lifts $\widetilde{\gamma} \looparrowright \mathbb{R}^2 \smallsetminus \mathbb{Z}^2$



- γ is called <u>linear</u> if $\widetilde{\gamma}$ can be isotoped so that $\widetilde{\gamma}'(t)$ is arbitrarily close to constant for all t

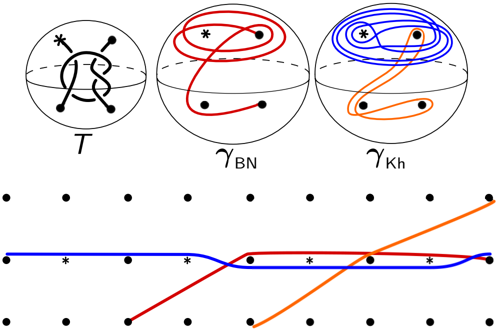
Theorem (K-Watson-Zibrowius'20; in progress)

If γ is a component of $\gamma_{\mathrm{Kh}}(T;\mathbb{F}_2)$, then γ is linear.

In other words, curves organize themselves along lines.

Examples + Proof via fishtails

Example: Curves in γ_{Kh} are linear; γ_{BN} is not.



Proof:

- Linearity $\iff$ No $> 360^\circ$ wrapping $\Longleftarrow$ $(\star)$ Fishtails cancel
- The proof of $(\star)$ is algebraic. Translating to symplectic geometry:
- $\gamma_{\mathrm{Kh}} = [\gamma_{\mathrm{BN}} \xrightarrow{H} \gamma_{\mathrm{BN}}]$ $\Longrightarrow$ γ_{Kh} is unobstructed in $\mathcal{S} \smallsetminus * = \mathcal{S}^*_4 \cup \therefore$, i.e. fishtails enclosing three punctures $\therefore$ cancel.

Unobstructedness in $\mathcal{S}_3$ via extension property

(proof continued)

We have: γ_{Kh} is unobstructed in $\mathcal{S}_4^* \cup \bullet\!:$ (Not true for γ_{BN})

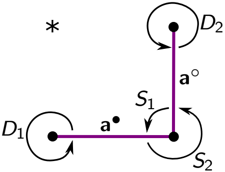
Goal: both $\gamma_{\mathrm{BN}}, \gamma_{\mathrm{Kh}}$ are unobstructed in $\mathcal{S}_3 = \mathcal{S}_4^* \cup *$

Idea: to prove that γ_{BN} is an object of $\mathcal{W}_{\mathrm{def}}(\mathcal{S}_3)$ deformed by the divisor $* = U^2$, i.e. to construct an extension

$$\boxed{Д(T)^{\mathcal{B}^*[U]}} \xrightarrow{U=0} Д(T)^{\mathcal{B}} \xrightarrow{[\mathrm{HKK}]} \gamma_{\mathrm{BN}}, \quad \text{where}$$

- $\mathcal{B} := \mathrm{End}_{\mathcal{W}(\mathcal{S}_4^*)}(\mathbf{a}^\circ \oplus \mathbf{a}^\bullet)$
- $\mathcal{B}^*[U] := \mathrm{End}_{\mathcal{W}_{\mathrm{def}}(\mathcal{S}_3)}(\mathbf{a}^\circ \oplus \mathbf{a}^\bullet)$ is the deformed A_∞ algebra,

e.g. $\mu_4(D_2, S_1, D_1, S_2) = U^2$ since the rectangle covers $*$ once.
[Haiden-Katzarkov-Kontsevich'14]



Connection to HMS

$$\underset{\textbf{(1)}}{M(T)^{\mathcal{A}}} \xrightarrow[\textbf{(2)}]{\text{(HMS)}} \boxed{Д(T)^{\mathcal{B}^*[U]}} \xrightarrow{U=0} Д(T)^{\mathcal{B}} \xrightarrow{\text{[HKK]}} \gamma_{\text{BN}}$$

Question: How to construct $Д(T)^{\mathcal{B}^*[U]}$? Two steps:

(1) using matrix factorization framework [Khovanov-Rozansky'08], construct a twisted complex $M(T)^{\mathcal{A}}$, where

$$\mathcal{A} = \text{End}\left(\mathbb{F}_2[x,y,z] \underset{yz}{\overset{x}{\rightleftarrows}} \mathbb{F}_2[x,y,z] \oplus \mathbb{F}_2[x,y,z] \underset{xz}{\overset{y}{\rightleftarrows}} \mathbb{F}_2[x,y,z] \right)$$

is a dg-enhancement of $\mathcal{B}$ (thanks to W. Ballinger)

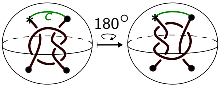
(2) obtain $Д(T)^{\mathcal{B}^*[U]}$ via the HMS $(\mathbb{A}^3, xyz) \longleftrightarrow \mathcal{S}_3$:

$$MF(\mathbb{A}^3, xyz) \underset{\simeq}{\overset{\text{[Orlov'04]}}{\longleftrightarrow}} D_{sg}(xyz=0) \underset{\simeq}{\overset{\left[\substack{\text{Abouzaid-Auroux-}\\ \text{Efimov-Katzarkov-Orlov'13}}\right]}{\longleftrightarrow}} \mathcal{W}(\mathcal{S}_3)$$

$\implies \mathcal{A} \simeq \mathcal{B}^*[U]/(U=1)$ (thanks to Y. Lekili)

Remark: Method from **(1)** provides a natural A_∞ structure on all arc algebras $\mathcal{H}_n$

Mutation questions



Conjecture 1.
$\mathrm{Kh}(K;\mathbb{Q})$ is preserved by mutation.

Theorem. (K-Watson-Zibrowius'19) Suppose a 4-ended tangle T has horizontal connectivity . Then, mutating tangle T preserves the underlying curves $\gamma_{\mathrm{BN}}(T), \gamma_{\mathrm{Kh}}(T)$, but changes the local system for each component γ by multiplying $\times(-1)^{\#\gamma\cap c}$.

Corollaries.

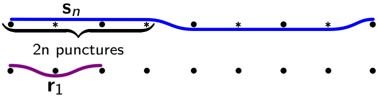
• In characteristic two, Bar-Natan homology $\widetilde{\mathrm{BN}}(K;\mathbb{F}_2)$ is preserved by mutation. (Known for $\mathrm{Kh}(K,\mathbb{F}_2)$ [Wehrli'10, Bloom'10])

• Rasmussen's invariant $s^{\mathbf{k}}(K)$ is preserved by mutation for any $\mathbf{k}$.

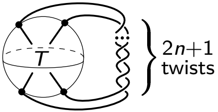
What is left to prove Conjecture 1:

• Pinning down signs in MCG naturality and geography results

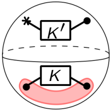
• Proving that all components of $\gamma_{\mathrm{Kh}}(T)$ look as follows: (up to MCG action)



The GCCC holds asymptotically



Family $\{K_n\}_{n\in\mathbb{Z}}$



Horizontally split.

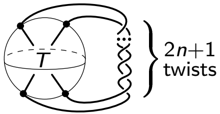
Conjecture 2. (Generalized Cosmetic Crossing Conjecture)
Assuming T is not horizontally split, all the knots in the family $\{K_n\}$ are different.

Question. What happens for $n \gg 0$? (J. Wang)

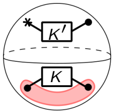
Theorem. (K-Lidman-Moore-Watson-Zibrowius'20; in progress)
If T is not horizontally split, then there exists N such that the knots $\{K_n\}_{|n|\geqslant N}$ are all different.

Main ingridients:
(1) comparing $\operatorname{rk}\widetilde{\operatorname{Kh}}(K_n)$ using immersed curves
(2) split-tangle detection result:

Split-tangle detection



Family $\{K_n\}_{n\in\mathbb{Z}}$



Horizontally split.

Theorem. (K-Lidman-Moore-Watson-Zibrowius'20; in progress) 4-ended tangle T is horizontally split $\iff$ $\gamma_{\mathrm{Kh}}(T)$ can be homotoped into the red neighborhood indicated above.

The proof is based on a spectral sequence between annular Khovanov and annular instanton Floer homologies [Xie'18].

Remark: Asymptotic GCCC cannot be proved using knot Floer homology.

Plan: Use gradings on $\widetilde{\mathrm{Kh}}$ to obtain stronger results towards GCCC.

Thank you!