Review and True/False questions, Bretscher 2.2-3.1

Review

2.2 Linear transformations in geometry.

Before proceeding, remember that we use notation $e_1, \ldots, e_n$ for the standard basis, i.e.

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Also, recall, that for any matrix $A_{m \times n}$ one has

$$A = \begin{bmatrix} | & & | \\ Ae_1 & \ldots & Ae_n \\ | & & | \end{bmatrix}$$

We covered what different 2x2 matrices mean geometrically, accompanying each case with illustrations.

1. *Scaling.* Matrix $A = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ corresponds to scaling the plane by a factor k. I.e. $A\vec{x} = k\vec{x}$ for any vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

2. *Projection.* Suppose $a^2 + b^2 = 1$. Then matrix $A = \begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix}$ corresponds a projection on the line $L = \{t \begin{bmatrix} a \\ b \end{bmatrix} | t \in \mathbb{R}\}$, which is the line "along" the vector $\begin{bmatrix} a \\ b \end{bmatrix}$.

3. *Reflection.* Suppose $a^2 + b^2 = 1$. Then matrix $A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ corresponds to a reflection about the line $L = \{t \begin{bmatrix} a \\ b \end{bmatrix} | t \in \mathbb{R}\}$, which is the line "along" the vector $\begin{bmatrix} a \\ b \end{bmatrix}$.

4. *Rotation.* Suppose $a^2 + b^2 = 1$ (one can write them as $a = \cos\theta$ and $b = \sin\theta$). Then matrix $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ corresponds to a counter-clockwise rotation

of the plane around the origin by the angle θ, which we denote by R_θ. Note, that $R_\theta(e_1) = Ae_1 = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$.

5. *Rotation+scaling.* For any a, b matrix $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = r \cdot \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ (where $r^2 = a^2 + b^2$ and $r \geqslant 0$) corresponds to a counter-clockwise rotation of the plane around the origin by the angle θ, composed with scaling by a factor r. Note, that $Ae_1 = \begin{bmatrix} a \\ b \end{bmatrix} = r \cdot \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$.

6. *Shears.* Matrices $S_h = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}, S_v = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$ correspond to horizontal and vertical sheers respectively. In the case of horizontal sheer, the image of y-axis is a line of a slope $1/k$, passing through origin. Points on x-axis are preserved by A.

2.3 Matrix multiplication

Definition 1. If $A_{k\times m} = \begin{bmatrix} | & & | \\ \vec{v}_1 & \dots & \vec{v}_m \\ | & & | \end{bmatrix}$ and $B_{l\times k} = \begin{bmatrix} - & \vec{w}_1 & - \\ & \dots & \\ - & \vec{w}_l & - \end{bmatrix}$ then

$$B_{l\times k} \cdot A_{k\times m} = \begin{bmatrix} - & \vec{w}_1 & - \\ & \dots & \\ - & \vec{w}_l & - \end{bmatrix} \cdot \begin{bmatrix} | & & | \\ \vec{v}_1 & \dots & \vec{v}_m \\ | & & | \end{bmatrix} = \begin{bmatrix} \vec{w}_1 \cdot \vec{v}_1 & & \vec{w}_1 \cdot \vec{v}_m \\ & \ddots & \\ \vec{w}_l \cdot \vec{v}_1 & & \vec{w}_l \cdot \vec{v}_m \end{bmatrix}$$

Important. Matrix multiplication corresponds to a composition of linear transformations. (illustration)

Properties of matrix multiplication:

1. $(AB)C = A(BC)$
2. $A(B + C) = AB + AC$
3. $(kA)B = k(AB)$
4. AB does not always equal to BA!

Practically, this means that one should treat multiplication of matrices as multiplication of numbers, except often $AB \neq BA$. (another key difference is that one cannot always divide by a matrix, see the next section).

Example 1. We illustrated $AB \neq BA$ on the following two matrices:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

First, we just multiplied them both ways and saw that the results are not the same. Second, we understood geometrically why $AB \neq BA$, by looking at the image of the frame (e_1, e_2) under the linear transformations AB and BA.

2.4 Inverse

Recall that identity matrix $Id_{n\times n} = I_n$ is a matrix which has 1's on its diagonal, and 0's everywhere else. This matrix can be characterized as the only linear transformation $A : \mathbb{R} \to \mathbb{R}$, s.t. $A\vec{x} = \vec{x}$ for any $\vec{x} \in \mathbb{R}$.

Definition 2. Linear transformation $T_1 : \mathbb{R}^n \to \mathbb{R}^n$ is invertible if it is invertible as a function, i.e. there is such a linear transformation $T_2 : \mathbb{R}^n \to \mathbb{R}^n$, that $T_2(T_1(\vec{x})) = \vec{x}$ for any $\vec{x} \in \mathbb{R}^n$.

Notice, that from this definition it is clear, that the matrix which represents composition $T_2 \circ T_1$ is I_n.

There is a corresponding notion of an inverse for matrices.

Definition 3. Matrix $B_{n\times n}$ is said to be inverse of the matrix $A_{n\times n}$, if $AB = I_n$.

Definition 4. Matrix $B_{n\times n}$ is said to be inverse of the matrix $A_{n\times n}$, if $BA = I_n$.

These definitions are equivalent. Thus, it is may be better to remember the following definition:

Definition 5. Matrix $B_{n\times n}$ is said to be inverse of the matrix $A_{n\times n}$, if $BA = AB = I_n$. In this case we denote $B = A^{-1}$.

If one has an equation $A\vec{x} = \vec{b}$, and A is invertible (i.e. has an inverse), then one can find a solution by multiplying both sides by A^{-1}:

$$A^{-1}A\vec{x} = A^{-1}\vec{b} \iff \vec{x} = A^{-1}\vec{b}$$

($\iff$ means implication in both directions, and reads as "equivalently")

Thus it is important to understand: 1) Does the matrix has an inverse? 2) How one can find an inverse?

Criterion. *A is invertible* $\iff$ *A is a square matrix of size* $n \times n$ *and it has the full rank* $rk(A) = n$. *This is equivalent to saying that* $rref(A) = I_n$, *because* $rk(A)$ *is, by definition, the number of leading ones in rref.*

Theorem (how to find inverse)**.** *If* $A_{n\times n}$ *is invertible, then one can use Gauss-Jordan elimination method to find* A^{-1}. *Namely, one has to find rref of* $n \times 2n$ *matrix* $[A|I_n]$*:*

$$rref\left(\left[\begin{array}{c|c} A & I_n \end{array}\right]\right) = \left[\begin{array}{c|c} I_n & A^{-1} \end{array}\right]$$

Note also, that if A and B are invertible matrices, we have $(AB)^{-1} = B^{-1}A^{-1}$. Note that the order of A and B changes.

3.1 Image and Kernel

This is one of the most important sections in the book, it is important to understand well these two concepts.

Suppose we have a linear transformation, given by a matrix $A_{n\times m}$:

$$A : \mathbb{R}^m \to \mathbb{R}^n$$

Definition 6. Image of A is a subset of a target $Im(A) \subset \mathbb{R}^n$, consisting of such elements $\vec{y} \in \mathbb{R}^n$, that $A\vec{x} = \vec{y}$ for some $\vec{x} \in \mathbb{R}^m$ in the domain.

Definition 7. Kernel of A is a subset of a domain $Ker(A) \subset \mathbb{R}^m$, consisting of such elements $\vec{x} \in \mathbb{R}^m$, that $A\vec{x} = 0$.

Important. Finding Kernel of A is the same as solving an equation $A\vec{x} = 0$.

Definition 8. Suppose we have vectors $v_1, v_2, \ldots, v_k \in \mathbb{R}^m$. Then $span(v_1, v_2, \ldots, v_k) \subset \mathbb{R}^m$ is the set of vectors consisting of all possible linear combinations $c_1v_1 + \cdots + c_kv_k$.

Statement. *Suppose* $A = \begin{bmatrix} | & & | \\ v_1 & \ldots & v_k \\ | & & | \end{bmatrix}$ *(by the way, remember that this means* $v_i = Ae_i$*, see the beginning of these notes). Then one has*

$$Im(A) = span(v_1, v_2, \ldots, v_k)$$

How to write Ker(A) as a span? Finding kernel is the same as solving $A\vec{x} = 0$. So just use the Gauss-Jordan elimination method to solve it, and in the end you will get a parameterization of the space of solutions by free variables. From this one can write Ker(A) as a span. (and therefore, as an image)

How to write Im(A) as a kernel? This one is trickier, we won't cover it, but it is extremely useful to know how to do that. Key idea: first find (write as a span or image) the orthogonal compliment of Im(A) by solving $A^T\vec{x} = 0$. Then one can write $Ker(A^T) = Im(B)$. Then $Ker(B)$ is the one you were looking for.

Theorem. *Suppose A is a square $n \times n$ matrix. Then the following are equivalent:*

1. *A is invertible.*
2. *$Ker(A) = 0$, i.e. the only vector sent to 0 by A is 0.*
3. *$Im(A) = \mathbb{R}^n$, i.e. the image is the whole $\mathbb{R}^n$.*
4. *$rank(A) = n$ (remember that this is equivalent to $rref(A) = I_n$).*
5. $\det(A) \neq 0$ *(this one we will cover later)*

True/False questions

First we covered a couple of T/F questions from previous review session (chapters 1.1-2.1 of the book) material.

Then we discussed what is "proof by contradiction", see Example 3 from "T:F_intro.pdf" from last time, or *wikipage*.

Warning: "proof by contradiction" is not the same as "proof by contrapositive", which is discussed in Appendix B of Bretscher. "Proof by contradiction" is a general strategy for proofs of any statements. "Proof by contrapositive" can only be applied to statements like "A => B". The strategy is to prove instead "not B => not A".

T/F questions for this time:

Problem 1. *There exists an invertible matrix* $A \neq I$ *such that* $A^2 = A^3$.

Solution. False. Let us prove it by contradiction. Suppose there is such an invertible matrix A. Then, by multiplying both sides of $A^2 = A^3$ by A^{-2} (here we use invertibility), one gets $I = A$. This contradicts the assumption $A \neq I$. Therefore such an invertible matrix does not exists. □

Problem 2. *Any reflection is invertible.*

Solution. True. Reflection is an inverse of itself. □

Problem 3. A *is* $n \times n$. *Then* $A^2 = 0$ *if and only if* $Im(A) \subseteq Ker(A)$

Solution. True. One has to write down the definition of Im(A) and Ker(A). □

Remark. Notice, that if one wants to prove statement like "A is if and only if B", then he really needs to prove two statements. One is "A => B", and another is "B => A".

Problem 4. *There is a* 2×2 *matrix* A *such that Im(A)=Ker(A).*

Solution. True. Matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is an example. □

Problem 5. *If* A *is* $n \times n$ *matrix with* $A^2 = 0$, *then* $I + A$ *is invertible.*

Solution. True. $I - A$ is the inverse. Also, one can show directly that there is no such $\vec{x} \neq 0$, that $(I + A)\vec{x} = 0$ (cause otherwise $A(\vec{x}) = -\vec{x}$ and $A^2\vec{x} = \vec{x}$). This means that Ker(I+A)=0, which means I+A is invertible by a theorem above. □

Problem 6. *If A,B are invertible and of the same size, then A+B is invertible. Harder version – assume also that A,B have no negative entries.*

Solution. False. Take $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. □

Review and True/False questions, Bretscher 3.2-3.3

Review

3.2 Subspaces, bases, linear dependence

Definition 1. Vectors $\vec{v}_1, \dots, \vec{v}_n$ are called linearly independent if there are no "relations" between them, i.e. there is no non-trivial linear combination $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$ which is equal to zero. In other words, if $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = 0$, then one has to have $c_1 = \cdots = c_n = 0$.

Example 1. $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ are lin. independent.

Example 2. $\begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ are lin. dependent.

Definition 2. Subspace of $\mathbb{R}^n$ is a subset $W \subset \mathbb{R}^n$ s.t.

1. $\vec{0} \in W$
2. $\vec{x} \in W \implies c\vec{x} \in W$ for any scalar c (i.e. real number). This is being closed under multiplication.
3. $\vec{x}, \vec{y} \in W \implies \vec{x} + \vec{y} \in W$. This is being closed under addition.

The point is that subspaces behave just like linear spaces $\mathbb{R}^k$, they just sit inside a bigger R^n. I

Example 3. We listed all the subspaces in $\mathbb{R}^1, \mathbb{R}^2$.

Example 4. Any $span(\vec{v}_1, \dots, \vec{v}_n)$, any $Im(A)$, and any $Ker(A)$ are subspaces.

Fact. Set of solutions to $A\vec{x} = \vec{b}$ is a subspace only if $\vec{b} = 0$ (illustrated on the blackboard).

Definition 3. Set of vectors $\{\vec{v}_1, \dots, \vec{v}_k\}$ is called a basis of a subspace $W \subset \mathbb{R}^n$ if

1. $span(\vec{v}_1, \dots, \vec{v}_n) = W$
2. $\vec{v}_1, \dots, \vec{v}_n$ are lin. independent

Every subspace has a basis. It is not unique (illustrated on the blackboard). But the number of vectors in the basis is unique, see the next section.

Important (finding bases of Im(A) and Ker(A)). We illustrated in details two methods below on a matrix

$$A = \begin{bmatrix} 2 & 4 & 6 & 18 \\ 1 & 2 & 4 & 3 \end{bmatrix}$$

1. How to find a basis of $span(\vec{v}_1, \ldots, \vec{v}_k)$? In other words, how to eliminate all redundant vectors? In other words, how to find basis of $Im(A)$, where $A = \begin{bmatrix} | & & | \\ \vec{v}_1 & \ldots & A\vec{v}_k \\ | & & | \end{bmatrix}$?
   The answer is to just run Gauss-Jordan elimination method to find $rref(A)$, and then pick into basis of $Im(A)$ those columns of A, which correspond to leading ones in $rref(A)$. This method works, because elementary row operations do not change relations between column vectors. The columns with leading ones are linearly independent by Theorem 3.2.5 from the Bretscher.

2. How to find a basis of $Ker(A)$? One again has to do Gauss-Jordan elimination. After that one has a parameterization of the space of solutions to $A\vec{x} = 0$ by free variables. Treating free variables as coefficients in the linear combination, one gets a presentation of $Ker(A)$ as a span of vectors. Those vectors will be the basis of $Ker(A)$.

Remark. These two methods of finding basis actually prove rank-nullity theorem, because $dim(Ker(A)) = nullity(A)$ is a number of free variables in $rref(A)$, and $dim(Im(A)) = rk(A)$ is the number of leading one in $rref(A)$.

3.3 Dimension of a subspace

Definition 4. Dimension of a subspace $W \subset \mathbb{R}^n$ is the least number of vectors in the basis of W.

Fact (definition of dimension is correct). Any subspace $W \subset \mathbb{R}^n$ has a basis. It is not unique, but the number of vectors in it is always the same.

Example 5. Find out what is the dimension of a line, plane, of our space, of the whole space-time universe ($\mathbb{R}^4$). What about general case of $\mathbb{R}^n$? (rmk: the dimension of $\mathbb{R}^0$ is 0)

Example 6. Find out what is the dimension of every subspace of $\mathbb{R}^2$, $\mathbb{R}^3$ (simultaneously recalling what are those subspaces). Notice, that it makes sence that subspace always has less dimension then the ambient space.

Now we are going to state theorems, which are important, and which we are going to use while solving T/F questions.

Theorem (how to tell if vectors are linearly independent). *Suppose we are given m vectors $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{R}^n$. Then the following are equivalent (i.e. if one of the statements is true, the others should also true):*

1. *$\vec{v}_1, \ldots, \vec{v}_m$ are linearly independent*

2. *there is no redundant vector in $\vec{v}_1, \ldots, \vec{v}_m$, i.e. $\vec{v}_1, \ldots, \vec{v}_m$ give a basis of $span(\vec{v}_1, \ldots, \vec{v}_m)$.*

3. *none of $\vec{v}_i$ is a linear combination of the rest of the vectors*

4. *Ker* $\begin{bmatrix} | & & | \\ \vec{v}_1 & \dots & \vec{v}_m \\ | & & | \end{bmatrix} = \{\vec{0}\}$

5. *rank* $\begin{bmatrix} | & & | \\ \vec{v}_1 & \dots & \vec{v}_m \\ | & & | \end{bmatrix} = m$

The next theorem is one of the main ones.

Theorem (rank-nullity theorem)**.**
(Number of columns of A)=dim(Ker(A)) + dim(Im(A)).

It is called rank-nullity theorem because dim(Ker(A)) is called nullity(A), and dim(Im(A))=rk(A). Let's illustrate rank-nullity theorem on goto matrices for reflection, projection, rotation.

Example 7. Reflection $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (general form is $A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ having $a^2 + b^2 = 1$). The rank is 2, while the nullity is 0.

Example 8. Projections: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The ranks are 1 and 2, the nullities are 1 and 1 respectively.

Example 9. Rotation $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (general form is $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$). The rank is 2, the nullity is 0.

This is a very useful fact to remember:

Fact (rank inequality for the product)**.**
For any two matrices (which one can multiply) the following inequalities are true:

$$rk(A \cdot B) \leqslant rk(A)$$

$$rk(A \cdot B) \leqslant rk(B)$$

Finally, let us repeat the invertibility criterion, now adding two more equivalent statements (last ones), and adding to the statement the fact that rk(A)=dim(Im(A)).

Theorem (different disguises of invertibility)**.** *Suppose A is a square $n \times n$ matrix. Then the following are equivalent:*

1. *A is invertible.*

2. *$Ker(A) = 0$, i.e. the only vector sent to 0 by A is 0.*

3. *$Im(A) = \mathbb{R}^n$, i.e. the image is the whole $\mathbb{R}^n$.*

4. $rk(A) = dim(Im(A)) = n$ *(remember that this is equivalent to* $rref(A) = I_n$*).*

5. $\det(A) \neq 0$ *(this one we will cover later)*

6. All rows of A are linearly independent.

7. All columns of A are linearly independent.

True/False questions

Problem 1. $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}$. If $\vec{v}_1, \vec{v}_2$ are linearly independent, and $\vec{v}_2, \vec{v}_3$, then $\vec{v}_1, \vec{v}_3$ are linearly independent.

Solution. False. Counterexample would be $\vec{e}_1, \vec{e}_2, \vec{e}_1$ on a plane. □

Problem 2. Set of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$ satisfying $x^2 = y^2$ is a subspace.

Solution. False. It is a union of lines $x = y$ and $x = -y$, which is not closed under addition. □

Problem 3. There is a 2×5 matrix A, s.t. $dim(Ker(A)) = 2$

Solution. False. Suppose it exists. From $dim(Ker(A)) = 2$ by rank-nullity theorem we have $rk(A) = 3$. But matrix with two rows cannot have rank more then two. This is contradiction, so such matrix cannot exist. □

Problem 4. a) There are 3×3 matrices A and B of rank 2, s.t. AB has rank 1.
b) There are 3×3 matrices A and B of rank 2, s.t. AB has rank 0.

Solution. a) True. Just take matrices which represent projections on x-y and y-x plane.
b) False. Suppose they exist. Then it means that AB is a matrix of zeroes, because it is the only matrix, whose rank is 0. This means that if $B = \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix}$, then $AB = \begin{bmatrix} | & | & | \\ A\vec{v}_1 & A\vec{v}_2 & A\vec{v}_3 \\ | & | & | \end{bmatrix}$, and so $A\vec{v}_1 = A\vec{v}_2 = A\vec{v}_3 = \vec{0}$. Thus one has

$$Im(B) \subset Ker(A)$$

We know that rk(B)=dim(Im(B))=2, and rk(A)=2. Thus by rank-nullity dim(Ker(A))=1. Thus from $Im(B) \subset Ker(A)$ we get that 2 dimensional space is a subspace of 1 dimensional space — that is impossible, and so we have a contradiction. Thus such matrices do not exist. □

Problem 5. If A is 4×2 matrix, and B is 2×4 matrix, then $nullity(AB) \geqslant 2$.

Solution. True. By rank-nullity $nullity(AB) \geqslant 2 \iff rk(AB) \leqslant 2$. And this follows from rank inequality $rk(AB) \leqslant rk(A) \leqslant 2$, because A has 2 columns. □

Problem 6. $V^2, W^2 \subset \mathbb{R}^4$ are 2d subspaces, s.t. $V \cap W = \{\vec{0}\}$. Suppose $\{\vec{v}_1, \vec{v}_2\}$ is a basis of V, and $\{\vec{w}_1, \vec{w}_2\}$ is a basis of W. Then $\{\vec{v}_1, \vec{v}_2, \vec{w}_1, \vec{w}_2\}$ is a basis of $\mathbb{R}^4$

Solution. True. Suppose $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{w}_1 + c_4\vec{w}_1 = \vec{0}$. Then $c_1\vec{v}_1 + c_2\vec{v}_2 = -c_3\vec{w}_1 - c_4\vec{w}_1$, and because $V \cap W = \{\vec{0}\}$, we get $c_1\vec{v}_1 + c_2\vec{v}_2 = -c_3\vec{w}_1 - c_4\vec{w}_1 = \vec{0}$. Because $\{\vec{w}_1, \vec{w}_2\}$ and $\{\vec{v}_1, \vec{v}_2\}$ are bases, we get that $c_1 = c_2 = c_3 = c_4 = 0$, q.e.d. □

Problem 7. Matrices A and B have the largest possible rank given their size (min(n,m) for $n \times m$ matrix). Then their product AB also has the largest possible rank given its size.

Solution. False. Take $A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $B = [1, 0]$, then $AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ □

Problem 8. Suppose A is 2×4, and B is 4×7 matrices. Then

a) set of vectors in $\mathbb{R}^4$ which are simultaneously in Ker(A) and Im(B) is a subspace.

b) set of vectors in $\mathbb{R}^4$ which are either in Ker(A) or Im(B) is a subspace.

Solution. a) True. Intersection of subspaces is always a subspace.

b) False. Union of subspace is not always a subspace. □

Problem 9. Suppose A is 3×2, and B is 2×3 matrices. Then it can happen that $AB = I_3$.

Solution. False. Rank inequality. □

Review and True/False questions, Bretscher 3.4,5.1

Review

3.4 Change of coordinates

First of all, recall that our linear spaces $\mathbb{R}^n$ always come with a distinguished standard basis

$$\left\{e_1 = \begin{bmatrix}1\\0\\0\\\vdots\\0\end{bmatrix}, e_2 = \begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}, \dots, e_n = \begin{bmatrix}0\\0\\\vdots\\0\\1\end{bmatrix}\right\}$$

Recall that $\mathbb{R}^n$ is a space of length n sequences $\begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix}$ of numbers, which we call vectors. For every vector one has $\vec{x} = \begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix} = x_1\vec{e}_1 + \cdots + x_n\vec{e}_n$, and thus one calls $x_1, \dots, x_n$ coordinates of vector $\vec{x}$ w.r.t. basis $\{\vec{e}_1, \dots, \vec{e}_n\}$. The main question of this section is "What happens if one would a pick a different basis of $\mathbb{R}^n$"?

Suppose $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ is a different (from standard) basis of $\mathbb{R}^n$. This, by definition, means that for every vector $\vec{x} \in \mathbb{R}^n$ we have a unique decomposition along the basis

$$\vec{x} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$$

We call numbers $c_1, \dots, c_n$ coordinates of vector $\vec{x}$ w.r.t. basis $\mathcal{B}$, and denote these coordinates by $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix}c_1\\\vdots\\c_n\end{bmatrix}$.

Remark. If one denotes standard basis by $\mathcal{E}$, then one actually has $[\vec{x}]_{\mathcal{E}} = \begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix} = \vec{x}$. So when we write a column of vectors, we always secretly mean that these are coordinates w.r.t. standard basis.

Example 1. Take $\mathcal{B} = \{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\}$. What are the coordinates $\begin{bmatrix}1\\1\end{bmatrix}_{\mathcal{B}}$, i.e. what are the coordinates of vector $\begin{bmatrix}1\\1\end{bmatrix}$ w.r.t. to a new basis ? What are the coordinates $\begin{bmatrix}-4\\0\end{bmatrix}_{\mathcal{B}}$? What are the coordinates $\begin{bmatrix}-2\\-1\end{bmatrix}_{\mathcal{B}}$?

Answer these questions by solving a linear system $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2$. Illustrate the answers geometrically on the plane.

Main formulas

1. Change of coordinates formula. Suppose $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis of $\mathbb{R}^n$, and $S = \begin{bmatrix} | & & | \\ \vec{v}_1 & \ldots & \vec{v}_n \\ | & & | \end{bmatrix}$. Then one has a formula, which relates coordinates of vectors w.r.t. standard basis $\{\vec{e}_1, \ldots, \vec{e}_n\}$ and coordinates of vectors w.r.t. new basis $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$:

$$S \cdot [\vec{x}]_{\mathcal{B}} = \vec{x}$$

Because of this formula we sometimes call such S change of basis (or change of coordinates) matrix.

Example 2. For the previous example change of basis matrix would be $S = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$

2. Fix a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$. So far we always had a matrix

$$A = \begin{bmatrix} | & & | \\ Ae_1 & \ldots & Ae_n \\ | & & | \end{bmatrix}$$

which represents transformation T. Turns out that this matrix represents T *with respect to standard basis*. What does it mean for matrix A' to represent T w.r.t. different basis $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$? This matrix should send $\mathcal{B}$-coordinates to $\mathcal{B}$-coordinates, i.e.: $A'[\vec{x}]_{\mathcal{B}} = [T(\vec{x})]_{\mathcal{B}}$. Thus, by definition, this matrix equals

$$A' = \begin{bmatrix} | & & | \\ [T(\vec{v}_1)]_{\mathcal{B}} & \ldots & [T(\vec{v}_n)]_{\mathcal{B}} \\ | & & | \end{bmatrix}$$

(in analogy with matrix A, where $Ae_i = [T(e_i)]_{\mathcal{E}}$). Turns out that one can compute the matrix A' from matrices A and change of basis matrix S in this way:

$$A' = S^{-1}AS$$

Example 3. Understand what are A and A' matrices for the previous example, if T is a reflection about $x = y$ line. First spend some time to get the answer geometrically, from the definition of A', and then do a reality check by checking $A' = S^{-1}AS$.

These are equivalent definitions:

Definition 1. Two $n \times n$ matrices A and A' are called similar, if $A' = S^{-1}AS$ for some invertible matrix S.

Definition 2. Two $n \times n$ matrices A and A' are called similar, if there is such a linear transformation T and two bases $\mathcal{C}$ and $\mathcal{B}$ (one of them can be standard), s.t. A represents T w.r.t. $\mathcal{C}$, and A' represents T w.r.t. $\mathcal{B}$.

Similarity is transitive equivalence, i.e. if A is similar to B, B is similar to C, then A is similar to C. Also notice that A is similar to itself (take S=I), and notice that if A is similar to B, then B is similar to A (take S to be S^{-1}).

Another important fact is that similarity preserves rank.

5.1 Orthogonal projections and orthonormal bases

Orthogonality

Definition 3. Angle between two vectors is defined up to sign by

$$angle(\vec{v}, \vec{w}) = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}| \cdot |\vec{w}|},$$

where at the top one has dot product, and in the bottom one has a product of lengths, where length is defined as usual $|\vec{x}| = \sqrt{x \cdot x} = \sqrt{x_1^2 + \cdots + x_n^2}$.

Definition 4. $\vec{v}$ is *perpendicular* to $\vec{w}$ if $\vec{v} \cdot \vec{w} = 0$. Notation $\vec{v} \perp \vec{w}$.

Definition 5. *Orthogonal compliment* of subspace $V \subset \mathbb{R}^n$ is a set $V^\perp$ of vectors which are perpendicular to every vector in V.

Proposition.

1. *$V^\perp$ is also a subspace of $\mathbb{R}^n$ (in the first place we had that V is a subspace).*
2. *$dim(V^\perp) = n - dim(V)$*
3. *$(V^\perp)^\perp = V$*
4. *$V \cap V^\perp = \vec{0}$*

Example 4. What is orthogonal compliment of line $x = y$ on the plane?

Definition 6. Vectors $\vec{v}_1, \ldots, \vec{v}_k$ are called *orthonormal* if they are pairwise orthogonal length 1 vectors. In other words:

$$\vec{v}_i \cdot \vec{v}_j = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$

Fact. Orthonormal vectors are always linearly independent. Thus any n orthonormal vectors form a basis of $\mathbb{R}^n$.

We call such bases *orthonormal basis.* They behave like standard basis, and are very important. One can think of such basis as the image of standard basis after applying rotation or reflection.

Any subspace $V \subset \mathbb{R}^n$ has an orthonormal basis (not unique), and later we will see how to find one.

Orthogonal projection

If V is a subspace of $\mathbb{R}^n$, then any $\vec{x} \in \mathbb{R}^n$ can be split into sum

$$\vec{x} = \vec{x}^{\|} + \vec{x}^{\perp}$$

where $\vec{x}^{\|} \in V$ is the parallel to V part, and $\vec{x}^{\perp} \in V^{\perp}$ is the perpendicular to V part. Notice, that since $\vec{x}, \vec{x}^{\|}, \vec{x}^{\perp}$ form a triangle with the right angle, one has inequalities of length: $|\vec{x}^{\perp}| \leqslant |\vec{x}|, |\vec{x}^{\|}| \leqslant |\vec{x}|$.

Definition 7. *Orthogonal projection* of vector $\vec{x} \in \mathbb{R}^n$ onto subspace V is the parallel part of the vector: $proj_V \vec{x} = \vec{x}^{\|}$.

Transformation $T(\vec{x}) = proj_V \vec{x}$ is linear, and thus has its matrix. We will have a formula

Formula for orthogonal projection. If $\{\vec{v}_1, \dots, \vec{v}_k\}$ is orthonormal basis of $V \subset \mathbb{R}^n$, then

$$proj_V \vec{x} = (\vec{x} \cdot \vec{v}_1)\vec{v}_1 + \dots + (\vec{x} \cdot \vec{v}_k)\vec{v}_k$$

If one is interested in the matrix P, which represent orthogonal projection, then the best way would be to compute images of standard basis $T(\vec{e}_i) = proj_V \vec{e}_i$ based on the formula above, and then use the fact that $P = \begin{bmatrix} | & & | \\ T(\vec{e}_1) & \dots & T(\vec{e}_n) \\ | & & | \end{bmatrix}$

Remark. Orthogonal projection matrix P (on a subspace V) is similar to a matrix with 1's and zeroes on the diagonal, of the following form: (number of ones is equal to rank(P)=dim(V)).=

$$\begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & 1 & & \\ & & & & 0 & \\ & & & & & 0 \end{bmatrix}$$

Why is this true?

True/False questions

The following problem is a typical example of true/false question on similarity of matrices.

Problem 1. Rotations by $\pi/2$ and $\pi/4$ are similar.

Solution. False. If they are similar, then $SR_{\pi/2} = R_{\pi/4}S$ for some invertible 2×2 matrix S. Suppose

$$S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then, multiplying with rotation matrices, one gets

$$\begin{bmatrix} b & -a \\ d & -c \end{bmatrix} = 1/\sqrt{2} \begin{bmatrix} a-c & b-d \\ c+c & d+d \end{bmatrix}$$

From this it is an exercise to prove that S has to be a zero matrix, which is not invertible. This is contradiction, and thus two rotations cannot be similar. □

Problem 2. If A and B are invertible $n \times n$ matrices, then AB is similar to BA

Solution. True, take $S = B$. □

Problem 3. If A and B are similar, then A+I and B+I are similar

Solution. True, if SA=BS, then S(A+I)=(B+I)S. □

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Problem 4. If F is a non-zero 2×2 matrix, s.t. $F\vec{x} = 0$ for every $\vec{v} \perp \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then F is a matrix of orthogonal projection.

Solution. False, take $F = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ □

Problem 5. If $A^2 = A$, then matrix A is an orthogonal projection onto a subspace of $\mathbb{R}^n$.

Solution. False, take $A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ □

Problem 6. Matrices $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ are similar

Solution. True, right down equation SA=BS, solve for $S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and find out that there is a solution where S is invertible. □

Problem 7. There are four orthonormal vectors in $\mathbb{R}^3$.

Solution. False, 4 vectors cannot be linearly independent in 3D. □

Problem 8. There exist four vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ in $\mathbb{R}^2$, s.t. $\vec{v}_i \cdot \vec{v}_j < 0$ for $i \neq j$.

Solution. False, because angles between them should be more then 90°. □

Problem 9. There exist four vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ in $\mathbb{R}^2$, s.t. $\vec{v}_i \cdot \vec{v}_j < 0$ for $i \neq j$.

Solution. False, because angles between them should be more then 90°. □

Problem 10. $T = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}$ is similar to a matrix of some orthogonal projection.

Solution. True. Solve $ST = PT$, where $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. (we picked P in that form because we know that T has rank 1, and so if there is similar projection matrix, it will have rank 1, and thus will be similar to P) □

Review and True/False questions, Bretscher 5.2-5.4

Review

5.2 Gram-Schmidt process, QR factorization

Gram-Schmidt process

Remark. We will do everything in the case where $V \subset \mathbb{R}^n$ is three-dimensional. Generalization to k-dimensional subspaces is straightforward.

Suppose $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ is a basis of 3D subspace $V^3 \subset \mathbb{R}^n$. Gram-Schmidt process is an algorithm, which allows one to change vector by vector in the basis, s.t. in the end one gets an orthonormal basis $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ of the same subspace $V^3 \subset \mathbb{R}^n$. The formulas are the following: (accompanied with geometric illustration on the blackboard)

$$\begin{cases} \vec{u}_1 = & \frac{\vec{w}_1}{|\vec{w}_1|} \\ \vec{u}_2 = & \frac{\vec{w}_2^\perp}{|\vec{w}_2^\perp|}, \text{ where } \vec{w}_2^\perp = \vec{w}_2 - (\vec{u}_1 \cdot \vec{w}_2)\vec{u}_1 \\ \vec{u}_3 = & \frac{\vec{w}_3^\perp}{|\vec{w}_3^\perp|}, \text{ where } \vec{w}_3^\perp = \vec{w}_3 - (\vec{u}_1 \cdot \vec{w}_3)\vec{u}_1 - (\vec{u}_2 \cdot \vec{w}_3)\vec{u}_2 \end{cases}$$

Notice, that one has to compute $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ in the order, because formula for $\vec{u}_2$ involves $\vec{u}_1$, and formula for $\vec{u}_3$ involves $\vec{u}_1, \vec{u}_2$.

QR factorization

Suppose we are given a matrix $A_{l\times k}$, whose columns are linearly independent (i.e. columns form basis of Im(A)). Then, using Gram-Schmidt process, one can obtain *QR factorization*

$$A_{l\times k} = Q_{l\times k} R_{k\times k}$$

where Q is a matrix whose columns form orthonormal basis of Im(A), and R is upper triangular matrix.

Columns of Q can be found by running G-S process on columns of A. For k=3 the formula for factorization is the following:

$$A = \begin{bmatrix} | & | & | \\ \vec{w}_1 & \vec{w}_2 & \vec{w}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ | & | & | \end{bmatrix} \cdot \begin{bmatrix} |\vec{w}_1| & \vec{u}_1 \cdot \vec{w}_2 & \vec{u}_1 \cdot \vec{w}_3 \\ 0 & |\vec{w}_2^\perp| & \vec{u}_2 \cdot \vec{w}_3 \\ 0 & 0 & |\vec{w}_3^\perp| \end{bmatrix} = QR$$

Suppose A is $n \times n$ square matrix of full rk(A)=n. Then in factorization $A = QR$ the Q matrix is orthogonal.

Formula for the projection matrix. Suppose $\{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_k\}$ is a basis of a subspace $V \subset \mathbb{R}^n$. Then if $A = \begin{bmatrix} | & & | \\ \vec{w}_1 & \dots & \vec{w}_k \\ | & & | \end{bmatrix}$, one has $V = Im(A)$. Suppose $A = QR$ is QR-factorization of A. Then the matrix $Q \cdot Q^T$ is a matrix of orthogonal projection on V=Im(A). In other words:

$$proj_{Im(A)}\vec{x} = QQ^T \cdot \vec{x}$$

Remark. One very useful (and easy to check) property of orthogonal projection matrices is that $P^2 = P$.

Definition 1. Suppose matrix $A = \begin{bmatrix} | & & | \\ \vec{v}_1 & \dots & \vec{v}_k \\ | & & | \end{bmatrix}$. Then transpose of it is a matrix $A^T = \begin{bmatrix} - & \vec{v}_1 & - \\ & \vdots & \\ - & \vec{v}_k & - \end{bmatrix}$. In other words transposing corresponds to reflection of the matrix about it's $-45°$diagonal (notice, that matrix doesn't have to be square).

Properties of transpose matrices.

1. $(A^T)^T = A$
2. $(AB)^T = B^T A^T$
3. $(A^{-1})^T = (A^T)^{-1}$
4. $rk(A^T) = rk(A)$
5. $Ker(A)$ is orthogonal compliment of $Im(A^T)$
6. (dot product) $\vec{v} \cdot \vec{w} = \vec{v}^T \cdot \vec{w} = \vec{w}^T \cdot \vec{v}$ (products of matrices)

Warning. $rref(A)^T \neq rref(A^T)$

Definition 2. Symmetric matrices are those, which satisfy $A = A^T$. (makes sense only for square matrices)

Example 1. Orthogonal projection matrices P are symmetric, because

$$P^T = (QQ^T)^T = (Q^T)^T Q^T = QQ^T = P$$

5.3 Orthogonal matrices

Definition 3. The following are four equivalent definitions of *orthogonal* square $n \times n$ matrix A:

1. $(A^T)A = I_n = A(A^T)$
2. $A^{-1} = A$

3. columns of A form orthonormal basis of $\mathbb{R}^n$
4. Matrix A preserves dot product, i.e. $\vec{x}\cdot\vec{y}$=$A\vec{x}\cdot\vec{A}y$ for any two vectors $\vec{x},\vec{y}\in\mathbb{R}^n$. This is equivalent to saying that A preserves lengths, i.e. $|A\vec{x}|=|\vec{x}|$ for any $\vec{x}\in\mathbb{R}^n$

Remark. The terminology is a bit confusing, so it is good to have a clear distinction in mind between "orthogonal projection matrices", which never have a full rank, and "orthogonal matrices", which always have a full rank.

Example 2. What are orthogonal 2×2 matrices? Turns out those are all rotation matrices, and rotation composed with reflection matrices.

Properties of orthogonal matrices.

1. Orthogonal matrices always have full rank.
2. A is orthogonal $\implies$ A^T is orthogonal
3. A is orthogonal $\implies$ $det(A)=\pm 1$
4. $rk(A^T)=rk(A)$
5. Orthogonal projection matrices are never orthogonal matrices

5.4 Least squares method

Suppose we are given a system $A\vec{x}=\vec{b}$ which doesn't have a solution. It means that the subspace $Im(A)$ misses the vector $\vec{b}$ (illustration on the blackboard).

Then the best thing one would hope is to find such $\vec{b}^*\in Im(A)$, that $dist(\vec{b}^*,\vec{b})$ is minimal, and then find its preimage $\vec{x}^*$. This $\vec{x}^*$ is called least squares solution. This is equivalent to condition

$$A\vec{x}^*=proj_{Im(A)}\vec{b}\quad(=\vec{b}^*)$$

Turns out that there is a very convenient way to find $\vec{x}^*$, namely one has to solve

$$A^TA=A^T\vec{b}$$

Example 3. Fitting quadric to four points on the plane (-1.8), (0,8), (1,4), (2,16) — example 4 from the book.

Example 4 (T/F question). We are looking for least squares solution to $A_{4\times 3}\vec{x}=\vec{b}$. We know that $Im(A)^\perp=span(\begin{bmatrix}1\\1\\1\\1\end{bmatrix})$, $\vec{b}=span(\begin{bmatrix}2\\2\\0\\0\end{bmatrix})$. Then the claim is that $A\vec{x}^*=\begin{bmatrix}1\\1\\2\\-2\end{bmatrix}$

Solution. False. Let's prove it by contradiction: suppose its true. Key observation is that because $A\vec{x}^*=proj_{Im(A)}\vec{b}$ one has $\vec{b}-A\vec{x}^*\perp Im(A)$. Thus one has to have $\vec{b}-A\vec{x}^*=k\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$, which is not true in this case — contradiction. □

True/False questions

Problem 1. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear transformation. Then if $m \neq n$, then T cannot preserve lengths.

Solution. False. Counterexample is an embedding of plane into 3D space, given by matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ □

Problem 2. Matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ represents orthogonal projection.

Solution. False. 1st way to see it: $A^2 \neq A$. 2nd way: the matrix has full rank. □

Problem 3. If A is 2×2 matrix, such that A^2 is orthogonal, then A is also orthogonal.

Solution. False. Counterexample is $\begin{bmatrix} 0 & 2 \\ 1/2 & 0 \end{bmatrix}$. $A^2 = I_2$ is orthogonal, whereas A is not, because doesn't preserve lengths.

How one would find this solution? A possible approach is to just take matrix A in the general form $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then write down what does it mean to have A^2 orthogonal, in those equations assume $a = -d$, and the solution will come out. □

Problem 4. If B is orthogona, A is similar to B, then A is orthogonal.

Solution. False. Take A to be matrix of linear transformation which swaps vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (what is this matrix?). Then we have $S^{-1}AS = B$, where $S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (B is a matrix representing the same linear transformation as A, but in basis of columns of S). Thus B is similar to A. B is clearly orthogonal, but A is not, because it doesn't preserve lengths (because it swaps $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$). Thus we found counterexample. □

Problem 5. Suppose K is 2×2 a matrix of a projection on x-axis, L is a matrix of a projection on y-axis 2×2. Then KL, $(KL)^2$, $(KL)^3$, ... are all different matrices.

Solution. False. Compute K and L by the formula for orthogonal projection and notice that all KL, $(KL)^2$, $(KL)^3$, ... are zero matrices. □

Problem 6. A is similar to $A^T \implies$ A is symmetric.

Solution. False. Write down general equation for 2×2 case $SA = A^T S$, and see that there is a lot of freedom in choosing coefficients of S and A, allowing A being not symmetric. □

Problem 7. Suppose $\{\vec{v}_1, \ldots, \vec{v}_6\}$ is a basis of $\mathbb{R}^6$. By Gram-Schmidt one gets orthonormal basis $\{\vec{u}_1, \ldots, \vec{u}_6\}$. Suppose $V = span(\vec{v}_1, \ldots, \vec{v}_5)$, $\vec{x} = proj_V \vec{u}_6$. Then it is possible that $|\vec{x}| \geqslant |\vec{v}_1|/\sqrt{5}$

Solution. False. From the way G-S process works one gets $V = span(\vec{v}_1, \ldots, \vec{v}_5) = span(\vec{u}_1, \ldots, \vec{u}_5)$. Thus $\vec{u}_6 \perp V$. Thus $\vec{x} = proj_V \vec{u}_6 = \vec{0}$, and its length is of course 0. □

Problem 8. There is 3 matrix such that $Ker(A) = Im(A)$

Solution. False. From rank-nullity theorem one gets that $dim(Ker(A)) = dim(Im(A)) = 3/2$, which is not possible. □