1. Overview

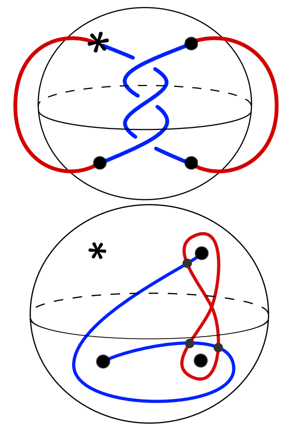


Figure 1

Over the last 20 years, numerous homological invariants of knots and 3-manifolds have been discovered. Heegaard Floer homology proved to be particularly powerful, due to its close connections with symplectic topology. My work is centered around the idea of applying similar symplectic techniques in Khovanov and instanton homology theories.

Specifically, I use methods from bordered Heegaard Floer theory and Fukaya categories of surfaces to study how tangle replacements affect knots and their invariants. A central achievement of my postdoctoral studies is Theorem 1 below, obtained in joint work with Watson and Zibrowius [KWZ19]. There, the topological input is a Conway sphere decomposition, that is a knot K and a two-sphere $\mathcal{S}$ intersecting the knot in four points, as on the top of Figure 1. In this setup, we obtain a symplectic geometric description of Khovanov homology in terms of Lagrangian Floer theory of curves in $\mathcal{S} \smallsetminus 4\mathrm{pt}$, as illustrated on the bottom of Figure 1. Much of my past and present research is focused on applying this result to various problems in knot theory:

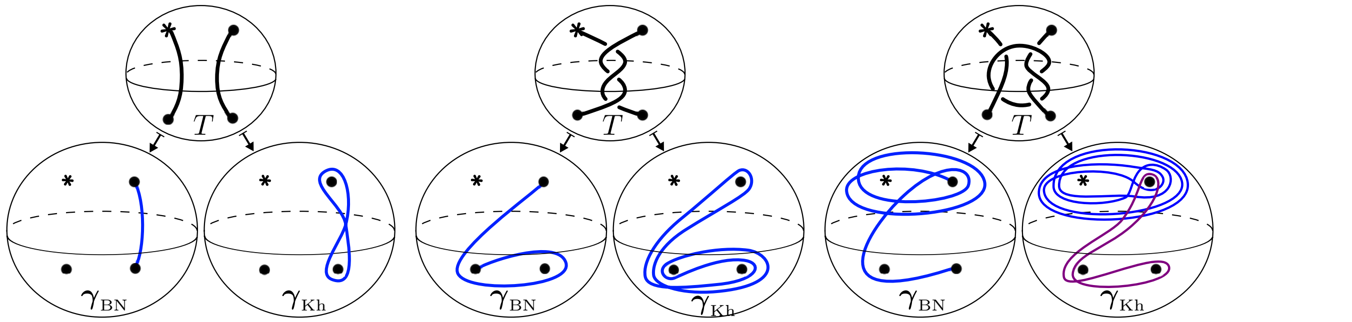
- *Mutation questions.* It was conjectured in [Lew09] that Rasmussen's s-invariant in preserved by mutation. We proved this result in [KWZ19], using structural properties of the wrapped Fukaya category of a surface. We also proved mutation invariance of Bar-Natan homology with coefficients in $\mathbb{F}_2$.
- *Thinness property of knots.* In [KWZ20c], given a Conway sphere decomposition, we provide a geometric criterion for when the knot is Khovanov-thin in terms of the two four-ended tangles. A key result in this project is based on the homological mirror symmetry statement for the three-punctured sphere [AAE+13].
- *Crossing change questions.* In [KLM+20], we prove the asymptotic version of the Generalized Cosmetic Crossing Conjecture. Namely, in the context of Figure 5a, given that the equator on the Conway sphere is not compressible, the knots K_n are all different for n large enough. The proof is based on the connection between Khovanov and instanton Floer homologies.

2. Novel Floer theoretic approach to Khovanov homology

Khovanov homology is a combinatorially defined knot invariant, taking the form of a bigraded vector space $\mathrm{Kh}(K)$ over an arbitrary field $\mathbf{k}$ [Kho00]. There is a slight variation called *reduced* Khovanov homology $\widetilde{\mathrm{Kh}}(K)$. To give several examples, for the unknot $\widetilde{\mathrm{Kh}}($$) = \mathbf{k}$, for the trefoil $\widetilde{\mathrm{Kh}}($$) = \mathbf{k}^3$, and for the figure-eight knot $\widetilde{\mathrm{Kh}}($$) = \mathbf{k}^5$. One of the major applications of Khovanov homology is Rasmussen's construction of the integer-valued concordance invariant $s(K)$. It was used to give the first combinatorial proof of Milnor's conjecture [Ras10], which states that the unknotting number of the (p,q)-torus knot is $\frac{1}{2}(p-1)(q-1)$.

A four-ended tangle T is a proper embedding of a pair of arcs into a three-ball: $T\colon\ I \sqcup I \hookrightarrow D^3$. Such objects arise naturally: any Conway sphere $\mathcal{S}$ intersecting a knot $K \subset S^3$ in four points decomposes the knot into two four-ended tangles $K = T_1 \cup_{\mathcal{S}} T_2$. An example of such a decomposition is depicted on the top of Figure 1, where K is the trefoil knot, the four-ended tangle T_1 consists of two red arcs outside the two-sphere, and the four-ended tangle T_2 consists of two blue arcs inside the two-sphere.

In [BN05], Bar-Natan defined a Khovanov-theoretic invariant of tangles, in the form of a chain complex $[\![T]\!]_{/l}$ over a certain category. One of the main results of our work [KWZ19] is a geometric interpretation of $[\![T]\!]_{/l}$, in case T has four ends. Figure 2 below provides a quick impression of the resulting tangle invariants, in the form of immersed curves. In more detail, suppose a tangle $T \subset D^3$ is *pointed*, which means that one of the four ends is marked by $*$. Denote by $\mathcal{S}^*_4$ the pointed



(a) Trivial tangle. (b) Three-twist tangle. (c) (2,-3)-pretzel tangle.

Figure 2. Curve invariants associated to three different tangles.

four-punctured sphere $\partial D^3 \smallsetminus \partial T$ coming from the boundary of the tangle. The claim is that $[\![T]\!]_{/l}$ can be interpreted as an immersed curve $\boldsymbol{\gamma}_{\mathrm{BN}}(T) \looparrowright \mathcal{S}^*_4$, considered up to homotopy. Furthermore, there is another immersed curve $\boldsymbol{\gamma}_{\mathrm{Kh}}(T) \looparrowright \mathcal{S}^*_4$ which interprets a slightly weaker version of $[\![T]\!]_{/l}$. A technical remark is that curves are bigraded in an appropriate sense, and may also carry local systems.

Examples of these curve-valued tangle invariants are depicted in Figure 2. The first one, in Figure 2a, is a trivial tangle: its $\boldsymbol{\gamma}_{\mathrm{BN}}$ invariant is an embedded arc, and its $\boldsymbol{\gamma}_{\mathrm{Kh}}$ invariant is a figure eight. The same is true for the second example, the three-twist tangle. Notice that in both cases, the curve $\boldsymbol{\gamma}_{\mathrm{BN}}(T)$ can be obtained from the tangle $T\colon I^* \sqcup I \hookrightarrow D^3$ by pushing the unmarked component I into the boundary sphere, in such a way that it does not intersect the other component I^*. This behavior generalizes to arbitrary *rational* tangles, i.e. those tangles that can be obtained from the trivial tangle (Figure 2a) by adding braid twists to the four punctures. An example of a non-rational tangle is depicted in Figure 2c, and the curve invariants in this case are more complicated. In particular, $\boldsymbol{\gamma}_{\mathrm{BN}}(T)$ is an immersed arc, and $\boldsymbol{\gamma}_{\mathrm{Kh}}(T)$ has two components.

In general, the curve $\boldsymbol{\gamma}_{\mathrm{BN}}(T)$ always contains exactly one arc, and for complicated tangles it may also have additional compact components. The curve $\boldsymbol{\gamma}_{\mathrm{Kh}}(T)$ always consists of only compact components. As for the relationship between $\boldsymbol{\gamma}_{\mathrm{BN}}(T)$ and $\boldsymbol{\gamma}_{\mathrm{Kh}}(T)$, in case of rational tangles, $\boldsymbol{\gamma}_{\mathrm{Kh}}(T)$ is obtained from $\boldsymbol{\gamma}_{\mathrm{BN}}(T)$ by substituting an arc with a figure eight. In general, the relationship is more complicated (see Figure 2c), and the precise geometric description of it is work in progress.

Khovanov homology is defined using planar diagrams of knots, and so it is natural to ask whether different, geometric techniques can be applied to studying Khovanov homology. The curve invariants $\boldsymbol{\gamma}_{\mathrm{BN}}$ and $\boldsymbol{\gamma}_{\mathrm{Kh}}$ make this possible via the following gluing result.

Theorem 1 ([KWZ19])**.** *Suppose a knot K is decomposed along a pointed Conway sphere, that is $K = T_1 \cup_{\mathcal{S}^*_4} T_2$. Then its reduced Khovanov homology (as a bigraded $\mathbf{k}$-vector space) is isomorphic to Lagrangian Floer homology of the curves associated to the two tangles:*

$$\widetilde{\mathrm{Kh}}(K) \cong \mathrm{HF}\left(\boldsymbol{\gamma}_{\mathrm{Kh}}(T_1), \boldsymbol{\gamma}_{\mathrm{BN}}(T_2)\right)$$

(In practice, the dimension of Lagrangian Floer homology is almost always equal to the minimal intersection number of curves.) Theorem 1 is illustrated in Figure 1, where the decomposition of the trefoil is depicted on the top, and the resulting Lagrangian Floer intersection picture is depicted on the bottom. Minimal intersection number 3 confirms the well-known fact that the reduced Khovanov homology of the trefoil is 3-dimensional: $\mathrm{HF}(\boldsymbol{\gamma}_{\mathrm{Kh}}(T_1), \boldsymbol{\gamma}_{\mathrm{BN}}(T_2)) \cong \mathbf{k}^3 = \widetilde{\mathrm{Kh}}($$)$.

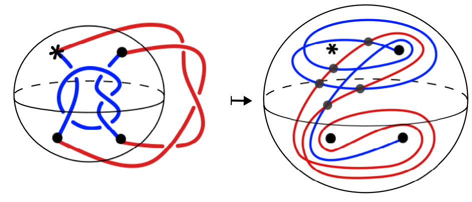


Figure 3. $\widetilde{\mathrm{Kh}}($ (-3,4)-torus knot $) = \mathbf{k}^5$.

Another example is depicted in Figure 3, where the Conway sphere decomposition of the (-3,4)-torus knot results in 5-dimensional Lagrangian Floer homology.

The full version of Theorem 1 includes similar gluing formulas for the unreduced Khovanov homology $\mathrm{Kh}(K)$ and reduced Bar-Natan homology $\widetilde{\mathrm{BN}}(K)$ (also known as equivariant Khovanov homology). In the latter case one has to work with *wrapped* Floer homology, since $\widetilde{\mathrm{BN}}(K)$ is always infinite dimensional.

To put Theorem 1 in context, the question of finding geometric origins of Khovanov homology has been a topic of intense study. There is Witten's gauge-theoretic proposal [Wit16], Cautis-Kamnitzer's algebro-geometric framework [CK08], and a series of papers of Seidel, Smith, Manolescu and Abouzaid [SS06, Man06, AS16, AS19] culminating in a symplectic version of Khovanov homology (based on the Floer theory of Hilbert schemes of Milnor fibers). The latter symplectic description is more general than Theorem 1, in the sense that the number of intersections $\mathcal{S} \cap K = 2k$ is not assumed to be 4. The advantage of our immersed curves framework is that the Floer theory happens in a surface. This makes intersection pictures concrete and computable, and ultimately leads to new insight on and applications of Khovanov homology.

Let me also describe the key ideas behind Theorem 1. Instrumental in the construction of $\boldsymbol{\gamma}_{\mathrm{BN}}(T)$ and $\boldsymbol{\gamma}_{\mathrm{Kh}}(T)$ is the following striking coincidence: Bar-Natan's universal cobordism category $\mathrm{Cob}_{/l}$ (restricted to diagrams with four ends) from [BN05] is quasi-equivalent to the wrapped Fukaya category $\mathcal{W}(\mathcal{S}^*_4)$. Another ingredient is a theorem of Haiden-Katzarkov-Kontsevich [HKK17], which classifies twisted complexes (or, equivalently, type D structures) over $\mathcal{W}(\mathcal{S}^*_4)$ in terms of immersed curves in $\mathcal{S}^*_4$. We reproved their theorem geometrically, which made the curve invariants $\boldsymbol{\gamma}_{\mathrm{BN}}(T)$, $\boldsymbol{\gamma}_{\mathrm{Kh}}(T)$ computable in practice [CZ], and we further classified morphism spaces between twisted complexes in terms of wrapped Floer homology of curves. The latter classification is the key behind the gluing result.

For applications below, the following mapping class group naturality result is important. Any braid twist of the four-punctured sphere defines a mapping class $\sigma \in \mathrm{MCG}(\mathcal{S}^*_4)$. Such a mapping class acts both on tangles and curves, and the claim is that over $\mathbb{F}_2$ this action commutes with curve-invariants: $\boldsymbol{\gamma}_{\mathrm{BN}}(\sigma(T); \mathbb{F}_2) = \sigma\left(\boldsymbol{\gamma}_{\mathrm{BN}}(T; \mathbb{F}_2)\right)$ and $\boldsymbol{\gamma}_{\mathrm{Kh}}(\sigma(T); \mathbb{F}_2) = \sigma\left(\boldsymbol{\gamma}_{\mathrm{Kh}}(T; \mathbb{F}_2)\right)$.

3. First application of Theorem 1: Mutation questions

Conway mutation is an operation on knots. Starting with a four-ended tangle decomposition $T_1 \cup_{\mathcal{S}} T_2$ (e.g. the top of Figure 1), mutation consists of cutting out the tangle T_2, rotating it by 180° degrees about the vertical axis, and gluing it back:

$$T_1 \cup_{\mathcal{S}} T_2 \xrightarrow{\text{Conway mutation}} T_1 \cup_{\mathcal{S}} \rotatebox{180}{T_2}$$

Mutation is known to be a subtle operation preserving many strong invariants, including hyperbolic volume [Rub87] and δ-graded knot Floer homology [Zib19].

Theorem 2 (Mutation invariance of $s^{\mathbf{k}}(K)$ and $\widetilde{\mathrm{BN}}(K; \mathbb{F}_2)$ [KWZ19])**.**
(1) *Khovanov and Bar-Natan homologies* $\mathrm{Kh}(K; \mathbb{F}_2)$, $\widetilde{\mathrm{Kh}}(K; \mathbb{F}_2)$, $\widetilde{\mathrm{BN}}(K; \mathbb{F}_2)$ *over the field of two elements are mutation invariant.*
(2) *Rasmussen's concordance invariant* $s^{\mathbf{k}}(K)$ *is preserved under mutation, for any field* $\mathbf{k}$.

The proof of (1) fulfills the original vision of Bar-Natan [BN], and is much shorter compared to the previously known strategies for $\mathrm{Kh}(K; \mathbb{F}_2)$ [Weh10, Blo10]. The mutation invariance of $\widetilde{\mathrm{BN}}(K; \mathbb{F}_2)$ is new. The main idea for the proof is a certain global way to move the basepoint $* \in \mathcal{S}^*_4$ from one puncture to another, based on the $4Tu$ relation in $\mathrm{Cob}_{/l}$.

The proof of (2) is based on certain structural properties of the wrapped Fukaya category $\mathcal{W}(\mathcal{S}^*_4)$, and uses our immersed curves framework in an essential way. Namely, the two key facts are the following: (a) mutation of T does not change the underlying curve $\boldsymbol{\gamma}_{\mathrm{BN}}(T)$, but rather multiplies the local system of $\boldsymbol{\gamma}_{\mathrm{BN}}(T)$ by ± 1; (b) to compute Rasmussen's s-invariant of a knot $K = T_1 \cup_{\mathcal{S}} T_2$, one only needs to know the arc components of $\boldsymbol{\gamma}_{\mathrm{BN}}(T_1)$ and $\boldsymbol{\gamma}_{\mathrm{BN}}(T_2)$, and arcs may only carry trivial local systems.

The most intriguing question remains to be the following conjecture:

Conjecture 3. *Khovanov homology over the field of rationals* $\mathrm{Kh}(K; \mathbb{Q})$ *is mutation invariant.*

This is our next project, and the strategy for proving the conjecture is already in place. Important in the proof will be pinning down signs in various results that we proved over $\mathbb{F}_2$: the aforementioned mapping class group naturality, as well as Theorem 5 below. The final piece will be resolving Conjecture 6, proof of which is expected to bear similarities with Theorem 5.

4. Second application of Theorem 1: Thinness property of knots

Throughout this section the coefficients are restricted to the two-element field $\mathbf{k} = \mathbb{F}_2$.

A knot K is called *Kh-thin* if $\widetilde{\mathrm{Kh}}(K)$ is concentrated along a single diagonal $const = q/2 - h$ with respect to quantum and homological gradings. All knots satisfy $\operatorname{rk}\widetilde{\mathrm{Kh}}(K) \geqslant \det(K)$, while for thin knots $\operatorname{rk}\widetilde{\mathrm{Kh}}(K) = \det(K)$, so one should think of the thinness property as a Khovanov analogue of being a Heegaard Floer L-space for 3-manifolds. That is, morally a knot is Kh-thin if Khovanov homology is as simple as possible.

A well-known result of Conway states that rational tangles are in one-to-one correspondence with $\mathbb{Q}P^1 = \mathbb{Q} \cup \{\infty\}$. Given an arbitrary four-ended tangle T, consider a set $\Theta(T) \subset \mathbb{Q}P^1$ of rational tangles R such that $T \cup_{\mathcal{S}} R$ is Kh-thin. If $\Theta(T)$ is an interval, denote by $\mathring{\Theta}(T)$ the interior of $\Theta(T)$.

Theorem 4 (Characterization of Kh-thin knots in the presence of Conway sphere [KWZ20c])**.**
(1) $\Theta(T)$ *is either empty, a single point, two points, or an interval inside* $\mathbb{Q}P^1$*.*
(2) Given a decomposition $K = T_1 \cup_{\mathcal{S}} T_2$*, if* $\mathring{\Theta}(T_1) \cup \mathring{\Theta}(T_2) = \mathbb{Q}P^1$ *then* K *is Kh-thin.*

This result is in direct analogy with the L-space gluing theorem [HRW16], which confirms the parallel between thin knots and L-spaces. The full version of Theorem 4 includes a more technical statement in the backwards direction of (2), as well as characterization of HFK-thin knots, where HFK-thinness is defined using knot Floer homology. Interestingly, the results point to the fact that thinness does not depend neither on the homology theory, nor on the field of coefficients.

The proof of Theorem 4 is based on the following result, which greatly restricts possible curves appearing as components of $\boldsymbol{\gamma}_{\mathrm{Kh}}(T)$. Given a curve $\gamma \looparrowright \mathcal{S}^*_4$, denote by $\widetilde{\gamma} \looparrowright \mathbb{R}^2 \smallsetminus \mathbb{Z}^2$ an arbitrary lift of γ to the cover $(\mathbb{R}^2 \smallsetminus \mathbb{Z}^2) \to (T^2 \smallsetminus 4\mathrm{pt}) \to \mathcal{S}^*_4$, where the second map is a quotient by the elliptic involution. A lift $\widetilde{\gamma}$ is called *linear* if it can be homotoped into a small neighborhood of a line in $\mathbb{R}^2$, such that the derivative $\widetilde{\gamma}'(t)$ stays in a small neighborhood of some slope p/q. See Figure 4 for examples of linearized lifts.

Theorem 5 (Geography of components of $\boldsymbol{\gamma}_{\mathrm{Kh}}(T)$ [KWZ20c])**.**
(1) If γ *is a component of* $\boldsymbol{\gamma}_{\mathrm{Kh}}(T)$*, then* γ *is linear.*
(2) If γ *is a component of* $\boldsymbol{\gamma}_{\mathrm{Kh}}(T)$ *and the linearized lift* $\widetilde{\gamma}$ *touches the special puncture* $*$*, then, after an appropriate reparameterization of* $\mathcal{S}^*_4$*, the curve* $\widetilde{\gamma}$ *looks as* $\mathbf{s}_n$ *from Figure 4 for some* $n \geqslant 1$*.*

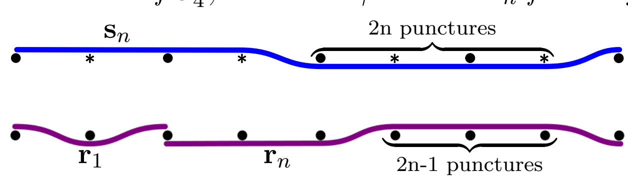


Figure 4. Curve families $\{\mathbf{s}_n\}^{n\geqslant 1}$ and $\{\mathbf{r}_n\}^{n\geqslant 1}$.

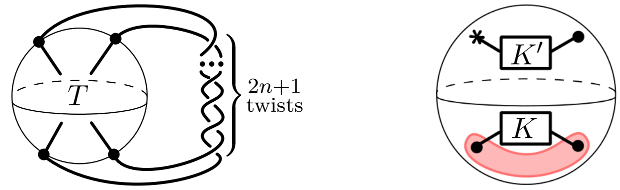
Looking at the curve $\boldsymbol{\gamma}_{\mathrm{Kh}}$ from Figure 2c, its more complicated top component is an example of $\mathbf{s}_2$ projected down to $\mathcal{S}^*_4$. To prove the linearity statement (1), we had to pin down the precise connection between Khovanov-Rozansky's matrix factorization framework [KR08] and Bar-Natan's cobordism category theory [BN05]. Interestingly, along the way we had to use deep results from derived algebraic geometry, such as Orlov's matrix factorization model of the category of singularities [Orl04], and homological mirror symmetry for the three-punctured sphere [AAE+13].

We are working on a similar statement for curves that do not touch the special puncture:

Conjecture 6. *If* γ *is a component of* $\boldsymbol{\gamma}_{\mathrm{Kh}}(T)$ *and the linearized lift* $\widetilde{\gamma}$ *does not touch the special puncture* $*$*, then, after an appropriate reparameterization of* $\mathcal{S}^*_4$*, the curve* $\widetilde{\gamma}$ *looks as* $\mathbf{r}_n$ *from Figure 4 for some* $n \geqslant 1$*.*

The figure eight curve from Figure 2a is an example of $\mathbf{r}_1$ projected down to $\mathcal{S}_4^*$. Conjecture 6 is the last meaningful step towards mutation invariance of $\mathrm{Kh}(K;\mathbb{Q})$; the rest will be a matter of pinning down the signs. We expect the proof of the conjecture to be similar to Theorem 5.

5. Third application of Theorem 1: Cosmetic crossings



(a) Family $\{K_n\}^{n\in\mathbb{Z}}$. (b) Horizontally split tangle.

Figure 5

The following two conjectures are currently being actively explored (see for example [LM17, Wan20]).

Conjecture 7 (Cosmetic Crossing Conjecture, #1.58 on Kirby's problem list [Kir97])**.** *Suppose T is a four-ended tangle consisting of two arcs connecting the top (bottom) ends to each other, and assume that T is <u>not</u> horizontally split (Figure 5b). Then the two knots $T\cup_{\mathcal{S}}$ ⤫ and $T\cup_{\mathcal{S}}$ ⤬ are different.*

Conjecture 8 (Generalized Cosmetic Crossing Conjecture)**.** *Provided T satisfies conditions of Conjecture 7, the knots $\{K_n\}^{n\in\mathbb{Z}}$ from Figure 5a are all different.*

The following result proves the second conjecture "asymptotically", i.e. for n large enough.

Theorem 9 ([KLM+20])**.** *Provided T satisfies conditions of Conjecture 7, there exists N such that the knots $\{K_n\}^{|n|\geqslant N}$ from Figure 5a are all different.*

The proof is based on comparing the ranks of $\widetilde{\mathrm{Kh}}(K_n)$ using Theorem 1, together with the following split-tangle detection result.

Theorem 10. *A four-ended tangle T is horizontally split if and only if the curve $\boldsymbol{\gamma}_{\mathrm{Kh}}(T)$ can be homotoped into a neighborhood indicated in Figure 5b.*

This, in turn, is proved using Xie's annular instanton Floer detection result [Xie18, Theorem 1.5].

Partial results in the direction of Theorem 9 can be obtained using classical methods, via Thurston's hyperbolic Dehn Surgery Theorem. Another remark is that Theorem 9 cannot be proved using knot Floer homology, which underpins the advantage of Khovanov homology in tackling tangle replacements questions. At the moment, we are exploring how gradings on $\widetilde{\mathrm{Kh}}(K_n)$ can further help with the Generalized Cosmetic Crossing Conjecture. In particular, we expect that geography results (Theorem 5 and Conjecture 6) should allow restricting the number of isotopic knots in $\{K_n\}^{n\in\mathbb{Z}}$ to 2. Another result we aim towards is a proof of the Cosmetic Crossing Conjecture for Khovanov-thin knots.

6. Other projects

In a joint work with Cazassus, Herald and Kirk [CHKK20], we computed the traceless $SU(2)$-character variety of the tangle $Q\subset S^2\times[0,1]$ depicted on the right. The character variety $R(Q)$ is defined as the space of representations modulo conjugation, see Equation (1) below. There are two restriction maps $R(S^2,4\mathrm{pt})\leftarrow R(Q)\rightarrow R(S^2,4\mathrm{pt})$ that correspond to inclusion maps $(S^2\times\{0\},4\mathrm{pt})\rightarrow(S^2\times[0,1],Q)\leftarrow(S^2\times\{1\},4\mathrm{pt})$. The traceless character

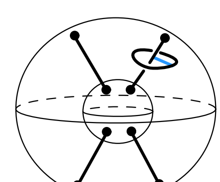


Figure 6. Tangle Q.

variety $R(S^2, 4\text{pt})$ is a pillowcase P—a two-sphere with four $\mathbb{Z}/2$-orbifold points, obtained as a torus modulo the elliptic involution.

$$(1)\quad R(Q) = \left\{\rho\colon\ \pi_1(S^2 \times [0,1] \smallsetminus Q) \to SU(2) \;\middle|\; \begin{smallmatrix}\rho \text{ is } -1 \text{ around the short blue arc in Q}\\ \rho \text{ traceless around the other strands of Q}\end{smallmatrix}\right\} \Big/ \text{conjugation}$$

Theorem 11.
(1) *$R(Q)$ is a singular 3-manifold, and there exists a particular holonomy perturbation such that the perturbed space $R_\pi(Q)$ is a genus three surface.*
(2) *The restriction map to the two pillowcases $R_\pi(Q) \to P^- \times P$ misses the orbifold singularities, and is a smooth Lagrangian immersion.*

Our initial motivation for computing $R_\pi(Q)$ was the pillowcase homology framework, which is the symplectic side of the Atiyah-Floer conjecture for singular instanton knot homology. But later we discovered a more striking consequence of Theorem 11: the computation of the map $R_\pi(Q) \looparrowright P \times P$ implies that the *figure eight bubbling*, a certain degeneration phenomenon predicted by Bottman and Wehrheim [BW18], does appear in the context of traceless character varieties.

Let me end with two smaller projects that were carried out recently. In a semi-expository paper [KWZ20a], my co-authors and I give an immersed curve interpretation of knot Floer homology over the ring $\mathbf{k}[U, V]/(UV)$. As an application, we take the result of Lipshitz, Ozsváth and Thurston [LOT18] that connects the bordered invariant of the knot exterior $\widehat{\mathrm{CFD}}(\mathring{\nu}(K))$ with knot Floer homology $\mathrm{CFK}^-(K)$, and interpret it as an elementary topological operation on immersed curves.

In another project [KWZ20b], we proved that two invariants of four-ended tangles, $\boldsymbol{\gamma}_{\mathrm{Kh}}(T)$ from [KWZ19] and L_T from [HHHK18], are equivalent to each other.

References

[AAE+13] M. Abouzaid, D. Auroux, A. I. Efimov, L. Katzarkov, and D. Orlov. Homological mirror symmetry for punctured spheres. *J. Amer. Math. Soc.*, 26(4):1051–1083, 2013. ArXiv: 1103.4322.

[AS16] M. Abouzaid and I. Smith. The symplectic arc algebra is formal. *Duke Math. J.*, 165(6):985–1060, 2016. ArXiv: 1311.5535.

[AS19] M. Abouzaid and I. Smith. Khovanov homology from Floer cohomology. *J. Amer. Math. Soc.*, 32(1):1–79, 2019. ArXiv: 1504.01230.

[Blo10] J. M. Bloom. Odd Khovanov homology is mutation invariant. *Math. Res. Lett.*, 17(1):1–10, 2010. ArXiv: 0903.3746.

[BN] D. Bar-Natan. An approach to mutation invariance of Khovanov homology. `http://drorbn.net/?title=Mutation_Invariance_of_Khovanov_Homology`.

[BN05] D. Bar-Natan. Khovanov's homology for tangles and cobordisms. *Geom. Topol.*, 9:1443–1499, 2005. ArXiv: math/0410495v2.

[BW18] N. Bottman and K. Wehrheim. Gromov compactness for squiggly strip shrinking in pseudoholomorphic quilts. *Selecta Math. (N.S.)*, 24(4):3381–3443, 2018. ArXiv: 1503.03486.

[CHKK20] G. Cazassus, C. M. Herald, P. Kirk, and A. Kotelskiy. The correspondence induced on the pillowcase by the earring tangle. 2020. ArXiv preprint 2010.04320.

[CK08] S. Cautis and J. Kamnitzer. Knot homology via derived categories of coherent sheaves. I. The sl(2)-case. *Duke Math. J.*, 142(3):511–588, 2008. ArXiv: math/0701194.

[CZ] G. Chhina and C. Zibrowius. Python module KhT.py for computing Khovanov-theoretic curve invariants of four-ended tangles. Available at `https://github.com/spinachstealer/KhT`.

[HHHK18] M. Hedden, C. M. Herald, M. Hogancamp, and P. Kirk. The Fukaya category of the pillowcase, traceless character varieties, and Khovanov cohomology, 2018. ArXiv preprint 1808.06957.

[HKK17] F. Haiden, L. Katzarkov, and M. Kontsevich. Flat surfaces and stability structures. *Publ. Math. Inst. Hautes Études Sci.*, 126:247–318, 2017. ArXiv: 1409.8611v2.

[HRW16] J. Hanselman, J. A. Rasmussen, and L. Watson. Bordered Floer homology for manifolds with torus boundary via immersed curves, 2016. ArXiv preprint 1604.03466v2.

[Kho00] M. Khovanov. A categorification of the Jones polynomial. *Duke Math. J.*, 101(3):359–426, 2000. ArXiv: math/9908171.

[Kir97] Problems in low-dimensional topology. In R. Kirby, editor, *Geometric topology (Athens, GA, 1993)*, volume 2 of *AMS/IP Stud. Adv. Math.*, pages 35–473. Amer. Math. Soc., Providence, RI, 1997.

[KLM+20] A. Kotelskiy, T. Lidman, A. H. Moore, L. Watson, and C. Zibrowius. Applications of Heegaard Floer and Khovanov curve-invariants of four-ended tangles. 2020. In preparation.

[KR08] M. Khovanov and L. Rozansky. Matrix factorizations and link homology. *Fund. Math.*, 199(1):1–91, 2008. ArXiv: math/0401268v2.

[KWZ19] A. Kotelskiy, L. Watson, and C. Zibrowius. Immersed curves in Khovanov homology. 2019. ArXiv preprint 1910.14584.

[KWZ20a] A. Kotelskiy, L. Watson, and C. Zibrowius. A mnemonic for the Lipshitz-Ozsváth-Thurston correspondence, 2020. ArXiv preprint 2005.02792.

[KWZ20b] A. Kotelskiy, L. Watson, and C. Zibrowius. Khovanov invariants via Fukaya categories: the tangle invariants agree, 2020. ArXiv preprint 2004.01619.

[KWZ20c] A. Kotelskiy, L. Watson, and C. Zibrowius. Thin links and Conway spheres. 2020. In preparation.

[Lew09] L. Lewark. The Rasmussen invariant of arborescent and of mutant links, 2009. Master thesis, ETH Zürich. http://www.lewark.de/lukas/Master-Lukas-Lewark.pdf.

[LM17] T. Lidman and A. H. Moore. Cosmetic surgery in L-spaces and nugatory crossings. *Trans. Amer. Math. Soc.*, 369(5):3639–3654, 2017. ArXiv: 1507.00699.

[LOT18] R. Lipshitz, P. S. Ozsváth, and D. P. Thurston. Bordered Heegaard Floer homology. *Mem. Amer. Math. Soc.*, 254(1216):viii+279, 2018. ArXiv: 0810.0687.

[Man06] C. Manolescu. Nilpotent slices, Hilbert schemes, and the Jones polynomial. *Duke Math. J.*, 132(2):311–369, 2006. ArXiv: math/0411015.

[Orl04] D. Orlov. Triangulated categories of singularities and D-branes in Landau-Ginzburg models. *Tr. Mat. Inst. Steklova*, 246(Algebr. Geom. Metody, Svyazi i Prilozh.):240–262, 2004. ArXiv: math/0302304.

[Ras10] J. Rasmussen. Khovanov homology and the slice genus. *Invent. Math.*, 182(2):419–447, 2010. ArXiv: math/0402131.

[Rub87] D. Ruberman. Mutation and volumes of knots in S^3. *Inventiones mathematicae*, 90(1):189–215, 1987.

[SS06] P. Seidel and I. Smith. A link invariant from the symplectic geometry of nilpotent slices. *Duke Math. J.*, 134(3):453–514, 2006. ArXiv: math/0405089.

[Wan20] J. Wang. The cosmetic crossing conjecture for split links, 2020. ArXiv preprint 2006.01070.

[Weh10] S. M. Wehrli. Mutation invariance of Khovanov homology over $\mathbb{F}_2$. *Quantum Topol.*, 1(2):111–128, 2010. ArXiv: 0904.3401.

[Wit16] E. Witten. Two lectures on gauge theory and Khovanov homology, 2016. ArXiv preprint 1603.03854.

[Xie18] Y. Xie. Instantons and annular Khovanov homology. 2018. ArXiv preprint 1809.01568.

[Zib19] C. Zibrowius. On symmetries of peculiar modules; or, δ-graded link Floer homology is mutation invariant, 2019. ArXiv preprint 1909.04267.